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The Field of a Uniformly Accelerated Current

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Abstract

The electromagnetic potentials of a long, straight wire carrying a steady current and being uniformly accelerated in a direction at right angles to the direction of the current flow are calculated exactly, and it is shown that the prediction by Cohn that there would be a non-zero component of the electric field parallel to the wire is incorrect.

In a paper published in this journal, Cohn (1969) derived approximate expressions for the electric field parallel to a long, straight wire carrying a steady current j under two assumptions: first, that the wire was being uniformly accelerated and was observed from an inertial frame in which it was instantaneously at rest, and secondly that the wire was being supported in a gravitational field. Cohn found that in the latter case the parallel component of the electric field would vanish but that in the former it would not, and deduced from this that the covariant formulation of electrodynamics in general relativity is incompatible with the principle of equivalence. The purpose of this note is to show that Cohn's result regarding the accelerated wire is wrong, and that consequently there is no such incompatibility.

Consider the following situation. An inertial observer views a straight wire of length l with its mid-point on the z-axis. The wire is being uniformly accelerated with proper acceleration g in the z-direction and at time $t = 0$ is at rest along the x-axis of the observer's coordinate system; at any other time t_Q the coordinates of a point on the wire will be

$$[x_Q, 0, \surd(\alpha^2 + c^2 t_Q{}^2) - \alpha] \qquad \alpha \equiv c^2/g \tag{1}$$

while the velocity of the wire will be

$$\mathbf{v} = \frac{c^2 t_Q}{\surd(\alpha^2 + c^2 t_Q{}^2)}\hat{\mathbf{z}} \tag{2}$$

Now consider an element of the wire dx_Q about the point whose coordinates are given by (1). Suppose that a signal emitted at time t_Q reaches a field point P with coordinates (x,y,z) at time t. Then the 'retarded time' t_Q can be determined by solving the equation

$$(x-x_Q)^2+y^2+[z-\sqrt{(\alpha^2+c^2t_Q^2)}+\alpha]^2=c^2(t-t_Q)^2 \tag{3}$$

the result being

$$ct_Q=\frac{ct\eta-\zeta\xi}{2(\zeta^2-c^2t^2)} \tag{4}$$

where

$$\eta=(x-x_Q)^2+y^2+\zeta^2+\alpha^2-c^2t^2$$
$$\zeta=z+\alpha$$
$$\xi=+[\eta^2-4\alpha^2(\zeta^2-c^2t^2)]^{1/2}$$

In its instantaneous rest frame† the element carries a steady current $\mathbf{j}'=j'\hat{\mathbf{x}}'$ so that the scalar and vector potentials at P due to the influence of this element are

$$\phi'=0, \qquad A_x'=\frac{j'}{c}\frac{dx_Q'}{r'}, \qquad A_y'=A_z'=0 \tag{5}$$

where r' is the distance from the element to P measured in the instantaneous rest frame of the element at the retarded time t_Q. When we make a Lorentz transformation to the inertial frame of the observer we find

$$j=j', \qquad dx_Q=dx_Q', \qquad \phi=0 \qquad \mathbf{A}=\mathbf{A}' \tag{6}$$

and since, as is well known, we can express r' in terms of unprimed quantities by the relation

$$r'=\frac{r-(1/c)\mathbf{v}.\mathbf{r}}{\sqrt{[1-(v^2/c^2)]}} \tag{7}$$

we have

$$\phi=0, \qquad A_x=\frac{j}{c}\frac{\sqrt{[1-(v^2/c^2)]}\,dx_Q}{r-(1/c)\mathbf{v}.\mathbf{r}}, \qquad A_y=A_z=0 \tag{8}$$

The quantities in (8) are all 'retarded' but we may express them in terms of (x,y,z,t) by using (4) and (2); after a straightforward calculation we obtain

$$\phi=0 \qquad \mathbf{A}=\left\{\frac{2j\alpha}{c\xi}dx_Q\right\}\hat{\mathbf{x}} \qquad \alpha\neq 0 \tag{9}$$

and since ξ is an even function of t we see that when $t=0$ then $\partial\mathbf{A}/\partial t=0$, and so the electric field due to the influence of the given element—and consequently that of the whole wire—vanishes exactly everywhere. This contradicts Cohn's result.‡

† Quantities measured in this frame will be denoted by a prime; unprimed quantities are measured in the rest frame of the inertial observer.

‡ If we compare Cohn's equation (A.3) with his final result (A.20) we see that the quantity within the brackets in (A.3) is of order $(1/c)$, and that consequently a term of order $(1/c)$ may not be neglected in evaluating it, as is done in (A.19).

We can also evaluate the magnetic field of the wire; the exact expressions lead to elliptic integrals, but if in the usual way we suppose the wire to be infinitely long (or, equivalently, suppose the field point P to be not too far from the wire) we can show that to order $(1/c^4)$:

$$H_x = 0$$

$$H_y = -\frac{2j}{c}\left\{\frac{z}{y^2+z^2} - \frac{1}{2}\frac{g}{c^2}\left[\frac{z^2}{y^2+z^2} + \log\sqrt{(y^2+z^2)}\right]\right\} \tag{10}$$

$$H_z = \frac{2j}{c}\left[\frac{y}{y^2+z^2} - \frac{1}{2}\frac{g}{c^2}\frac{yz}{y^2+z^2}\right]$$

This solution is, of course, restricted to the range $z + c^2/g + ct > 0$.

We have therefore the following result. If a long, straight, uniformly accelerating wire is viewed from an inertial frame in which it is instantaneously at rest, no electric field is observed. To order $(1/c^2)$ the magnetic field is that of a stationary wire, but an exact calculation reveals it to be somewhat different. It will be remarked that this is exactly analogous to the situation concerning a uniformly accelerated charge (Born, 1909, or see Fulton & Rohrlich, 1960): when viewed from an inertial frame in which the charge is momentarily at rest the magnetic field vanishes and the electric field is near Coulomb.

References

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